Math 279 Lecture 26 Notes

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1 Fixed Point Operators for Solving Abstract Regularity Structure PDEs

1.1 Fixed point operators for solving our ill-posed PDEs

We are interested in ill-posed problems like:

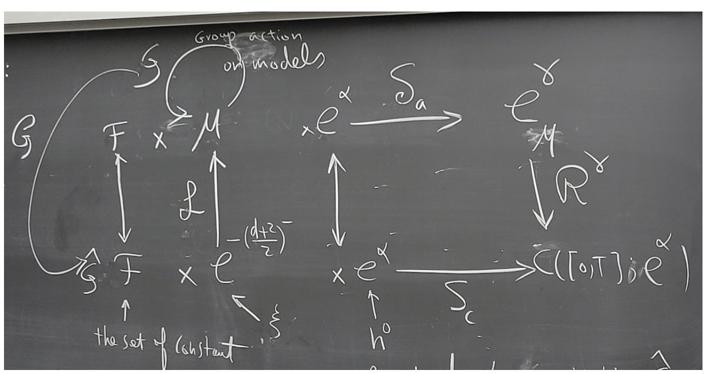
$$h_t = \Delta h + |h_x|^2 + \xi - C,$$

where ξ is white noise. This is subcritical iff $d \leq 1$. If we have

$$u_t = \Delta u - u^3 + \xi + C_1 + C_2 u,$$

this model is subcritical if $d \leq 3$. Here, we can vary the constants, so we are dealing with a family $\mathcal{F}$ of differential equations.

The general strategy for subcritical models is summarized in the following diagram:



We need to find a group action $\mathcal{G}$ on our model so that if $\xi^\varepsilon = \xi * \rho^\varepsilon$, then

$$\lim_{\varepsilon \to 0} \mathcal{S}_a M_\varepsilon(\mathcal{L}(\xi^\varepsilon))$$

exists, where M_ε is a suitable family of members of $\mathcal{G}$. This $\mathcal{G}$ would lead to a suitable $\widehat{\mathcal{G}}$ on $\mathcal{F}$. In our stochastic setting, since our distributions are all Gaussian, Wick's trick would allow us to discover what $\mathcal{G}$ is. Let us now focus on constructing $\mathcal{S}_a$ as we did last time.

As we discussed before, we consider the weak formulation

$$h_t = p * (|h_x|^2 + \xi) + \overline{h},$$

where p is the heat kernel and $\overline{h}$ solves the heat equation:

$$\begin{cases} \overline{h}_t = \overline{h}_{xx} \\ \overline{h}(x,0) = h^0(x). \end{cases}$$

For the other problem, we have

$$u = p * (-u^3 + \xi) + \overline{u}$$

with

$$\begin{cases} \overline{u}_t = \Delta \overline{u} \\ \overline{u}(x,0) = u^0(x). \end{cases}$$

Last time, we argued that $f \mapsto p * f$ can be lifted to a suitable operator $\mathcal{K}$ that can be decomposed as $\mathcal{K} = \mathscr{I} + \widehat{\mathcal{K}}$, where $\widehat{\mathcal{K}}$ is polynomial like annd $\mathscr{I}$ is somewhat local. Ideally, we could like to have this: An operator $\mathcal{K} : T \to T$ or $\mathcal{K} : \mathcal{C}^\alpha \to \mathcal{C}^{\alpha+2}$ so that

$$\begin{cases} \Pi_x(\mathcal{K}\tau) = p * \Pi_x \tau \quad (f \in \mathcal{C}^\alpha \ \Pi_x(\mathcal{K}f)(x) = p * \Pi_x f(x)) \\ \Gamma\mathcal{K}\tau = \mathcal{K}\Gamma\tau. \end{cases}$$

Such $\mathcal{K}$ would not exist. Here is the problem: if $\tau \in T_\alpha$, then $|(\Pi_x\tau)(\varphi_x^\delta)| \lesssim \delta^\alpha$. So $\mathcal{K}\tau \in T_{\alpha+2}$, and we must have an estimate of the form $|(\Pi_x(\mathcal{K}\tau))(\varphi_x^\delta)| \lesssim \delta^{\alpha+2}$. The problem is that in general, there is no reason for $p * \Pi_x\tau$ to vanish like $\delta^{\alpha+2}$ near the point x. This can be resolved if we subtract a suitable Taylor expansion. Motivated by this, we may define $\mathscr{I}$ by the following recipe. If $\tau \in T_\alpha$,

$$\Pi_x(\mathscr{I}\tau)(y) = p * \Pi_x\tau(y) - \sum_{k:|k|<\alpha+2} \frac{\partial^k(p * \Pi_x\tau)}{k!}(y-x)^k.$$

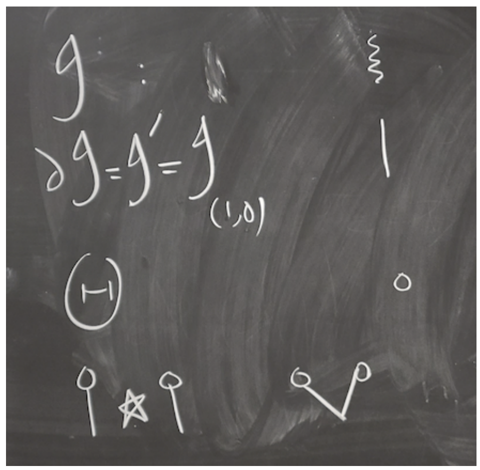
Because of this, we do not expect to have $\Gamma\mathscr{I}\tau = \mathscr{I}\Gamma\tau$, but we do have that $(\Gamma\mathscr{I} - \mathscr{I}\Gamma)(\tau)$ is in a sector of polynomials. (This should be compared with the differentiation operator: If "∂" is the lift of $\frac{\partial}{\partial x_1}$, then we do expect $\Pi_x(\partial\tau) = \frac{\partial}{\partial x_1}(\Pi_x\tau)$ and $\partial\Gamma = \Gamma\partial$.)

1.2 Using graphical notation with regularity structures to solve abstract PDEs

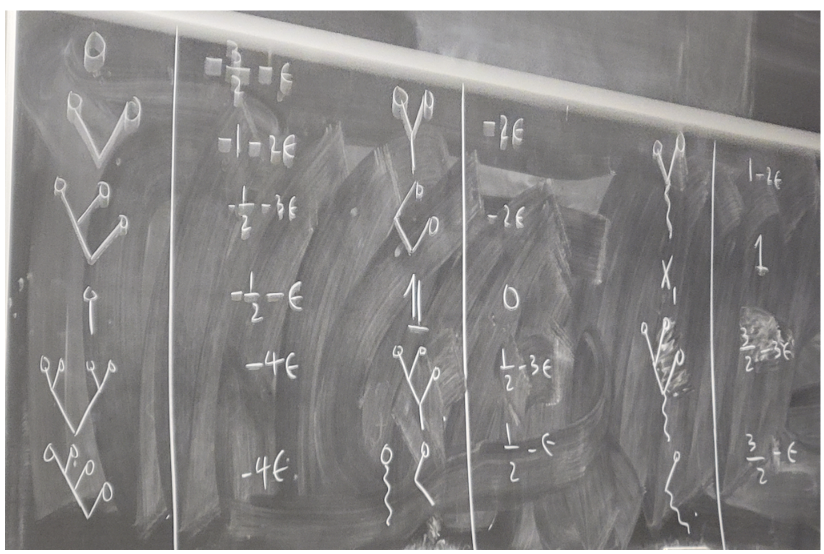
In the abstract version,

$$\begin{cases} H = \mathscr{I}((\partial H)^2 + \Xi) + \text{Polynomial part from } \widehat{K} + \overline{h}\mathbf{1} \\ \mathcal{U} = \mathscr{I}(\Xi - U^3) + \text{Polynomial part} + \overline{u}\mathbf{1}, \end{cases}$$

where Ξ represents white noise, we use the following graphical notation.[1]

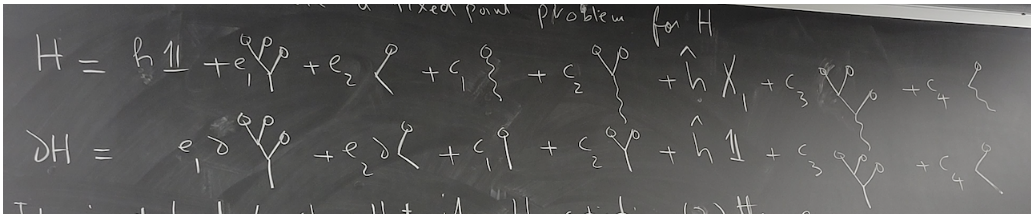


Here are all the terms we need to discuss H:

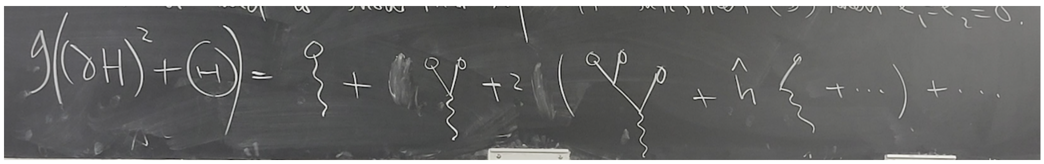


[1] I've decided to stop trying to type out any graphical notation. From here on out, it will all be pictures.

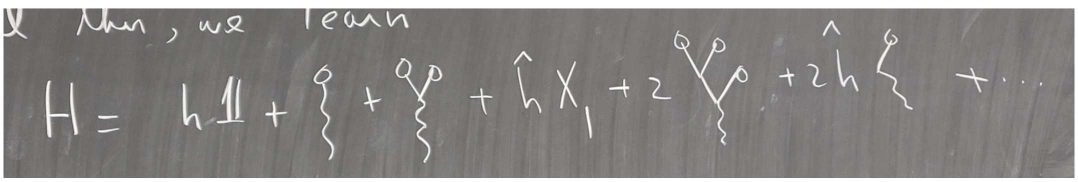
We want to formulate a fixed point problem for H.



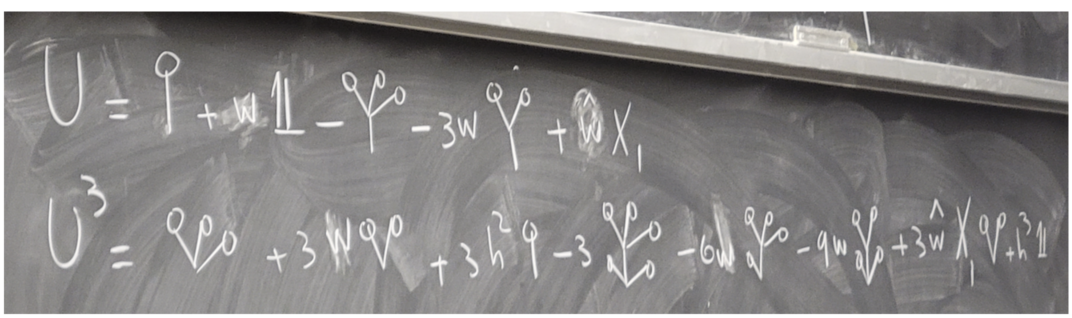
It can be shown that if H satisfies the abstract equation, then $e_1 = e_2 = 0$.



Here, we have noted that by comparing coefficients, we can see that $c_1 = c_2 = 1$, $c_3 = 2$, and $c_4 = \widehat{h}$. From all this, we learn that



We can play a similar game with the abstract equation for $\mathcal{U}$. To have simpler notation, we write $|$ for $\mathscr{I}$ (instead of $\wr$). We get



We still need to find the group G. This is a suitable set of transformations $\Gamma : T \to T$.

This group of 16×16 matrices is 7-dimensional.